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GENERALIZED HORIZONTAL-VERTICAL INTERPOLATION ARRAYS

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INTRODUCTION

The feasibility of using the summed output of a ring of regularly-spaced radially-oriented horizontal seismometers for estimating the output of a vertical seismometer located at the center of the ring was discussed in (Potter-Roden, 1965)* for an assumed noise model consisting of isotropic single-mode Rayleigh waves. The results of that preliminary study were encouraging enough to justify a deeper investigation of the theoretical noise-reduction capabilities of planar seismic arrays containing both horizontal and vertical instruments. In the present report we wish to consider the general problem of trying to use several concentric rings of horizontals to estimate the average output of a ring of verticals (concentric with the horizontal rings) in an isotropic noise field which may contain more than one mode. The present problem reduces to that considered in P-R if 1) there is only one horizontal ring, 2) the vertical ring has radius 0 and contains only one seismometer, and 3) there is only one significant noise mode.

The underlying motivation remains the same: the enhancement of a vertical or near-vertical P-wave. In the problem of trying to detect teleseismic events, and also in much oil exploration work, the "signal" is a vertical P-wave which may be deeply buried in surface wave noise. Let $v(t)$ be the average output, due to surface wave noise, of a ring of vertical seismometers, and let $h_1(t), \ldots, h_N(t)$ be the average outputs, due to surface wave noise, of N rings of horizontal seismometers. The functions $v(t)$ and $h_n(t)$ will always be assumed to be stationary time series. Since there is usually a degree of statistical coupling between horizontal and vertical components of surface wave noise, it is reasonable to try to design

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*Hereafter referred to as P-R

filters $g_n(t)$ to apply to h_n so as to make the autopower spectrum of

$$e(t) = v(t) - (g_1 \circledast h_1)(t) - \dots - (g_N \circledast h_N)(t) \tag{1.1}$$

as small as possible. On the other hand, since a vertical P-wave has no horizontal component, a processor of the type described by (1.1) will effect no distortion of the P-wave signal.

In this report we shall restrict our attention to processors of the type (1.1). For any such processor, we define the <u>interpolation error</u> to be

$$\text{I.E.} = E(f)/V(f) \tag{1.2}$$

where

E = autopower spectrum of $e(t)$

V = autopower spectrum of $v(t)$

If S_n = crosspower spectrum between h_n and v and C_{nj} = crosspower spectrum between h_n and h_j,

$$E = V - \sum_{n=1}^{N} G_n^* S_n - \sum_{n=1}^{N} G_n S_n^* + \sum_{n=1}^{N} \sum_{j=1}^{N} G_n^* C_{nj} G_j \tag{1.3}$$

where G_n = Fourier transform of g_n. The interpolation error is minimized if the frequency-domain filters G_n are chosen so as to satisfy the matrix equation

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_N \end{bmatrix} \tag{1.4}$$

Filters G_n satisfying (1.4) are called <u>optimum interpolation filters</u>, and for such a set of filters the interpolation error (1.2) becomes

$$\text{I.E.} = 1 - (1/V) \sum_{n=1}^{N} G_n S_n^* \tag{1.5}$$

<u>(Minimum interpolation error)</u>

All arrays considered in this report will have in common that the seismometers are deployed in concentric rings of regularly-spaced vertical or radially-oriented horizontal components. The restriction to this type of geometry has been based upon several considerations. First, the mathematical description of array response can be made more concise and understandable if the radial symmetry of the assumed noise field is complemented by corresponding symmetries in the array geometry; the optimum interpolation filters to be applied to the seismometers in a given ring are equal (or nearly so) for the arrays and noise fields which we shall study, and hence, we are justified in treating the average output of all seismometers in a ring as a single channel. Thus, if there are two rings of horizontals each containing six seismometers, we design only two filters, not twelve. Another consideration is that a ring average of regularly-spaced vertical or radially-oriented horizontal seismometers is fairly insensitive to directional characteristics of the noise field, provided the separation between seismometers is not large compared to wavelength; hence, although we assume isotropy throughout this investigation, the answers would be almost the same even if the noise field were assumed to be nearly unidirectional.

The use of rings of verticals rather than a single central vertical has a special purpose: a vertical ring average has a low response for certain wavelengths. This property can be used to overcome one of the serious problems connected with horizontal-vertical interpolation, namely that for higher modes there exist frequencies at which the horizontal motion vanishes, making interpolation of the vertical component impossible at those frequencies.

The first step in our investigation, the derivation of formulas for numerical calculation of ring crosspower spectra and optimum interpolation filters for a given array geometry and multimode surface wave noise field, is carried out in Section II. These same quantities may be expressed by means of different formulas, which are less useful for numerical calculation, but more interesting from the theoretical point of view because they give some insight into the difficult question of designing an array geometry to suit a particular noise field. These formulas are derived and discussed in Section III. Section IV describes and discusses a means for graphically representing the response of a given horizontal-vertical interpolation array to surface wave noise having arbitrary dispersion and horizontal-vertical coupling. Attention is also directed to the question of how the array responds to incident P-waves whose apparent angle of emergence is not quite vertical. Finally, in Section V, we present the results of numerical calculation of array response for several given array geometries and noise fields.

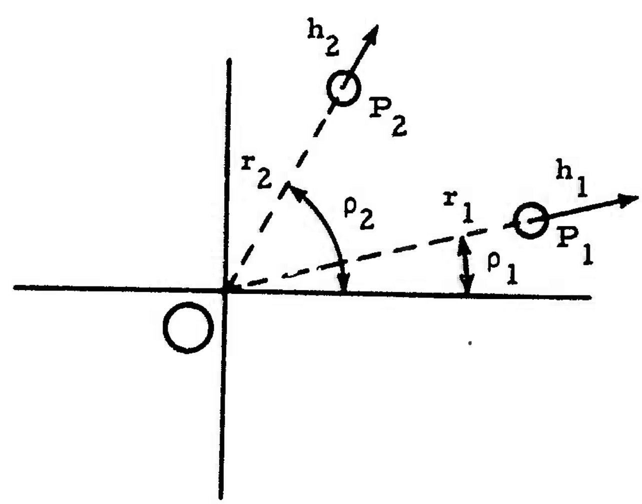
SECTION II

MULTIMODE RING CROSSPOWER SPECTRA AND OPTIMUM INTERPOLATION FILTERS

A. CROSSPOWER SPECTRUM FOR TWO SEISMOMETERS

Let O be the origin of polar coordinate system in the plane of the earth's surface. The position of a point P is specified by giving its polar coordinates (r, ρ).

At a given point $P_1 = (r_1, \rho_1)$ suppose we have a vertical seismometer v_1, and a horizontal seismometer h_1 which is radially oriented, i.e., the orientation angle of h_1 is ρ_1. At the point $P_2 = (r_2, \rho_2)$ let us be given a second pair of seismometers v_2 and h_2, where v_2 = vertical, h_2 = radially-oriented horizontal.



Now suppose, as in P-R, section 9, that we have an isotropic single-mode Rayleigh wave noise field, where the energy

propagates across the plane in pla .e waves and is uncorrelated from direction to direction. Let

$\Phi(f)$ = inline horizontal autopower spectrum

$K(f)$ = inline horizontal-vertical transfer function

$k(f)$ = wavenumber (cycles per unit distance)

We wish to obtain simple expressions for V = crosspower spectrum between v_1 and v_2, S = crosspower spectrum between h_1 and v_2, and C = crosspower spectrum between h_1 and h_2.

It may be shown (cf., P-R, section 9) that

$$V = |K|^2 \Phi \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(iu_2 \cos[\theta-\rho_2] - iu_1 \cos[\theta-\rho_1]\right) d\theta \tag{2.1}$$

$$S = -K\Phi \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\theta-\rho_1)\exp\left(iu_2 \cos[\theta-\rho_2] - iu_1 \cos[\theta-\rho_1]\right) d\theta$$

$$C = \Phi \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\theta-\rho_1)\cos(\theta-\rho_2)\exp\left(iu_2 \cos[\theta-\rho_2] - iu_1 \cos[\theta-\rho_1]\right) d\theta$$

where $u_1 = 2\pi k r_1$, $u_2 = 2\pi k r_2$. It is now useful to observe if we define

$$F(u_1, u_2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(iu_2 \cos[\theta-\rho_2] - iu_1 \cos[\theta-\rho_1]\right) d\theta$$

then formulas (2.1) become

$$V = |K|^2 \Phi \cdot F(u_1, u_2) \tag{2.2}$$

$$S = -iK\Phi \cdot \frac{\partial F(u_1, u_2)}{\partial u_1}$$

$$C = \Phi \cdot \frac{\partial^2 F(u_1, u_2)}{\partial u_2\, \partial u_1}$$

In order to obtain more convenient expressions for the crosspower spectra we note that

$$u_2 \cos(\theta-\rho_2) - u_1 \cos(\theta-\rho_1) = u \cdot \cos(\theta-\rho) \text{ where}$$

u and ρ are defined by

$$u = \left(u_1^2 + u_2^2 - 2u_1 u_2 \cos [\rho_1 - \rho_2]\right)^{\frac{1}{2}} \text{ and}$$

$$u \cos\rho = u_2 \cos\rho_2 - u_1 \cos\rho_1, \quad u \sin\rho = u_2 \sin\rho_2 - u_1 \sin\rho_1$$

Hence

$$F(u_1, u_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(iu \cos [\theta-\rho]\right) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(iu \cos\theta) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(u \cos\theta)\, d\theta$$

The last expression is a form of Poisson's integral and is equal to $J_0(u)$, where J_0 is the zero-th order Bessel function of the first kind (Watson, p. 47). Therefore,

$$F(u_1, u_2) = J_0(u) \tag{2.3}$$

where $u = \left(u_1^2 + u_2^2 - 2u_1 u_2 \cos [\rho_1 - \rho_2]\right)^{\frac{1}{2}}$

Substitution of (2.3) into the first formula of (2.2) yields

$$V = |K|^2 \Phi J_0(u) \tag{2.4}$$

In order to get the corresponding expressions for S and C we have to operate on (2.3) by taking the partial derivatives indicated in (2.2).

Using the standard Bessel function identities

$$J_n'(x) = \tfrac{1}{2}\left(J_{n-1}[x] - J_{n+1}[x]\right) \tag{2.5}$$

$$\frac{n}{x} J_n(x) = \tfrac{1}{2}\left(J_{n-1}[x] + J_{n+1}[x]\right)$$

$$J_{-n}(x) = (-1)^n J_n(x) \qquad \text{(Webster, p. 322)}$$

we find

$$\begin{aligned}\frac{\partial}{\partial u_1} F(u_1, u_2) &= \frac{\partial}{\partial u_1} J_0(u) = \frac{\partial u}{\partial u_1} \cdot J_0'(u) \\ &= -\left(\frac{u_1 - u_2\cos(\rho_1 - \rho_2)}{u}\right) \cdot J_1(u) \\ &= -\tfrac{1}{2}\left(u_1 - u_2\cos[\rho_1 - \rho_2]\right)\left(J_0[u] + J_2[u]\right)\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial u_2 \partial u_1} F(u_1, u_2) &= \frac{\partial}{\partial u_2}\left(-\frac{\partial u}{\partial u_1} J_1[u]\right) \\ &= \frac{-\partial^2 u}{\partial u_2 \partial u_1} J_1(u) - \frac{\partial u}{\partial u_1}\frac{\partial u}{\partial u_2} J_1'(u)\end{aligned} \tag{2.7}$$

$$= -\left(u\frac{\partial^2 u}{\partial u_2 \partial u_1}\right)\frac{J_1(u)}{u} - \frac{\partial u}{\partial u_1}\frac{\partial u}{\partial u_2} J_1'(u)$$

$$= -\tfrac{1}{2} u \frac{\partial^2 u}{\partial u_2 \partial u_1}\left(J_0[u] + J_2[u]\right) - \tfrac{1}{2}\frac{\partial u}{\partial u_1}\frac{\partial u}{\partial u_2}\left(J_0[u] - J_2[u]\right)$$

$$= -\tfrac{1}{2}\left(u\frac{\partial^2 u}{\partial u_2 \partial u_1} + \frac{\partial u}{\partial u_1}\frac{\partial u}{\partial u_2}\right) J_0(u) - \tfrac{1}{2}\left(u\frac{\partial^2 u}{\partial u_2 \partial u_1} - \frac{\partial u}{\partial u_1}\frac{\partial u}{\partial u_2}\right) J_2(u)$$

$$= -\tfrac{1}{2}\frac{\partial}{\partial u_2}\left(u\frac{\partial u}{\partial u_1}\right) J_0(u) - \tfrac{1}{2}\left[\frac{\partial}{\partial u_2}\left(u\frac{\partial u}{\partial u_1}\right) - 2\frac{\partial u}{\partial u_1}\frac{\partial u}{\partial u_2}\right]\cdot J_2(u)$$

$$= \tfrac{1}{2}\cos(\rho_1 - \rho_2) J_0(u) - \tfrac{1}{2}\left(u_1^2\cos[\rho_1 - \rho_2] + u_2^2\cos[\rho_1 - \rho_2] - 2u_1 u_2\right)\cdot\frac{J_2(u)}{u^2}$$

It may be verified from (2.5) that $\frac{J_2(u)}{u^2} = \frac{1}{8} J_0(u) + \frac{1}{6} J_2(u) + \frac{1}{24} J_4(u)$

Hence we may now collect our results in the equations

$$V = |K|^2 \Phi J_0(u) \tag{2.8}$$

$$S = iK\Phi \cdot \tfrac{1}{2}\left(u_1 - u_2\cos[\rho_1 - \rho_2]\right)\left(J_0[u] + J_2[u]\right)$$

$$C = \Phi \cdot \left\{\tfrac{1}{2}\cos(\rho_1 - \rho_2) J_0(u) - \left(u_1^2\cos[\rho_1 - \rho_2] + u_2^2\cos[\rho_1 - \rho_2] - 2u_1 u_2\right) \cdot\left(\frac{1}{16} J_0(u) + \frac{1}{12} J_2(u) + \frac{1}{48} J_4(u)\right)\right\}$$

where $u = \left(u_1^2 + u_2^2 - 2u_1 u_2 \cos[\rho_1 - \rho_2]\right)^{\frac{1}{2}}$. It may be shown that the formulas (2.8) reduce to the formulas 10.13 and 10.14 in P-R for the special case when $u_1 = u_2$.

B. SINGLE-MODE CROSSPOWER SPECTRUM FOR TWO RINGS OF SEISMOMETERS

We shall now compute the crosspower spectrum W for two concentric rings of regularly spaced seismometers. Thus let r_1, r_2, α_1, α_2 be real numbers, $r_1 \geq 0$, $r_2 \geq 0$, and let N_1 and N_2 be positive integers. Let

$$s_1(t) = (1/N_1)\left(a_1(t) + \ldots + a_{N_1}(t)\right) \quad \text{and}$$

$$s_2(t) = (1/N_2)\left(b_1(t) + \ldots + b_{N_2}(t)\right) \quad \text{where}$$

$a_m(t)$ is the output of a seismometer at the location $\left(r_1, \frac{2\pi m}{N_1} + \alpha_1\right)$ and $b_n(t)$ is the output of a seismometer at the location $\left(r_2, \frac{2\pi n}{N_2} + \alpha_2\right)$. It is assumed that either all the a_m's are vertical seismometers or they are all radially-oriented horizontals, and similarly for the b_n's. Thus $s_j(t)$ is the average output of a ring of N_j regularly spaced seismometers at distance r_j from the origin, with the ring having a rotation angle α_j, $j = 1, 2$. (Figure 2.1).

We wish to compute W, the crosspower spectrum between s_1 and s_2. If Φ_{mn} = the crosspower spectrum between a_m and b_n, then

$$W = \frac{1}{N_1 N_2} \cdot \sum_{m=1}^{N_1} \sum_{n=1}^{N_2} \Phi_{mn} \tag{2.9}$$

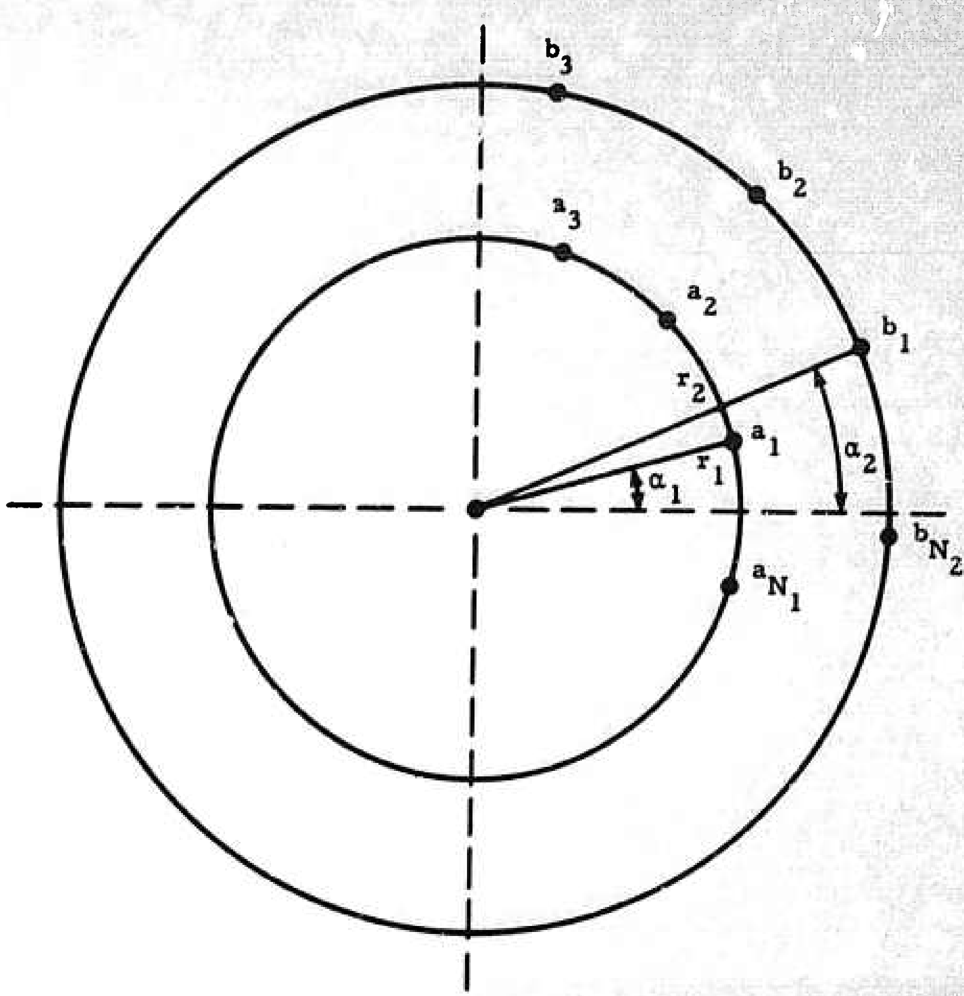


Figure 2.1. Schematic Diagram of Ring Array

Consider first the case when all the a_m's and b_n's are vertical seismometers. Then we have by (2.8) that the crosspower spectrum is

$$W = V_{1,2} = \frac{|K|^2 \Phi}{N_1 N_2} \cdot \sum_{m=1}^{N_1} \sum_{n=1}^{N_2} J_0(u_{mn}) \qquad (2.10)$$

(crosspower spectrum for two vertical ring averages)

where $u_{mn} = (u_1^2 + u_2^2 - 2u_1u_2\cos\phi_{mn})^{\frac{1}{2}}$, $u_1 = 2\pi kr_1$, $u_2 = 2\pi kr_2$,

and $$\phi_{mn} = \frac{2\pi m}{N_1} - \frac{2\pi n}{N_2} + \alpha_1 - \alpha_2$$

Similarly, if the a_m's are horizontals while the b_n's are verticals, we get from (2.8) that

$$W = S_{1,2} = \frac{iK\Phi}{2} \cdot \frac{1}{N_1N_2} \sum_{m=1}^{N_1} \sum_{n=1}^{N_2} \left\{ \left(u_1 - u_2\cos\phi_{mn}\right) \cdot \left(J_0[u_{mn}] + J_2[u_{mn}]\right)\right\} \tag{2.11}$$

(crosspower spectrum for a horizontal ring average and a vertical ring average).

Finally, if all the a_m's and b_n's are horizontals, we get

$$W = C_{1,2} = \Phi \cdot \frac{1}{N_1N_2} \cdot \sum_{m=1}^{N_1} \sum_{n=1}^{N_2} \left\{ \frac{1}{2}\cos\phi_{mn} J_0[u_{mn}] - \left(u_1^2\cos\phi_{mn} + u_2^2\cos\phi_{mn} - 2u_1u_2\right)\left(\frac{1}{16} J_0[u_{mn}] + \frac{1}{12} J_2[u_{mn}] + \frac{1}{48} J_4[u_{mn}]\right)\right\} \tag{2.12}$$

(crosspower spectrum for two horizontal ring averages).

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C. MULTIMODE CROSSPOWER SPECTRA

We shall now derive the array crosspower formulas for multimode noise. We suppose that the noise field has P modes, and that for the p^{th} mode,

$\Phi^{P}(f)$ = inline horizontal autopower spectrum
$K^{P}(f)$ = inline horizontal-vertical transfer function
$k^{P}(f)$ = wavenumber (cycles per unit distance),*

for p = 1,2, ...,P. We assume that any two modes are uncorrelated and that the energy from each mode is isotropic, uncorrelated from direction-to-direction, and propagates in plane waves across the half space. Our array consists of M rings of vertical seismometers and N rings of radially-oriented horizontal seismometers. In the m^{th} vertical ring we have A_m seismometers, located at the points whose polar coordinates are $(d_m, \frac{2\pi s}{A_m} + \alpha_m)$, $s = 1,2,\ldots,A_m$; the average output of the A_m seismometers in the m^{th} vertical ring is denoted by v_m. In the n^{th} horizontal ring we have B_n horizontal seismometers, located at the points $(r_n, \frac{2\pi s}{B_n} + \beta_n)$, $s = 1,2,\ldots B_n$; the average output of the B_n seismometers in the n^{th} horizontal ring is denoted by h_n. The numbers d_m and α_m are, respectively, the radius and the rotation angle of the m^{th} vertical ring, $m = 1,2,\ldots,M$; similarly, r_n and β_n are respectively the radius and rotation angle of the n^{th} horizontal ring, $n = 1,2, \ldots, N$.

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*In this report the symbol "p" appearing as a superscript is merely an index and is not an exponent. Any other superscripted symbol denotes an exponent.

Then according to (2. 10) and our assumption that there is no correlation between modes, we find that the crosspower spectrum V_{mn} between v_m and v_n is given by

$$V_{mn} = \frac{1}{A_m A_n} \sum_{p=1}^{P} \sum_{s=1}^{A_m} \sum_{t=1}^{A_n} |K^p|^2 \; \Phi^p \; J_0\left(u_{st}^{\;p}\right), \tag{2. 13}$$

(multimode vertical ring average crosspower spectrum)

where $u_{st}^{\;p} = 2\pi k^p (d_m^{\;2} + d_n^{\;2} - 2d_m d_n \cos\varphi_{st})^{\frac{1}{2}}$ and

$$\varphi_{st} = \frac{2\pi s}{A_m} - \frac{2\pi t}{A_n} + \alpha_m - \alpha_n .$$

If m = n, (2. 13) reduces to a slightly simpler formula for the vertical ring autopower spectrum $V_m = V_{mm}$:

$$V_m = \frac{1}{A_m} \sum_{p=1}^{P} \sum_{s=1}^{A_m} |K^p|^2 \; \Phi^p \; J_0\left(4\pi k^p d_m \sin\left[\frac{\pi s}{A_m}\right]\right) \tag{2. 14}$$

(multimode vertical ring average autopower spectrum).

Similarly, the crosspower spectrum S_{nm} between the average h_n of the n^{th} horizontal ring and the average v_m of the m^{th} vertical ring is obtained from (2. 11):

$$S_{nm} = \frac{1}{A_m B_n} \sum_{p=1}^{P} \sum_{s=1}^{A_m} \sum_{t=1}^{B_n} \frac{iK^p \Phi^p}{2} \cdot \left\{ 2\pi k^p \cdot \left(r_n - d_m \cos\varphi_{st}\right) \left(J_0\left[u_{st}^{\;p}\right] + J_2\left[u_{st}^{\;p}\right]\right)\right\} \tag{2. 15}$$

(multimode crosspower between horizontal and vertical ring averages)

where $\varphi_{st} = \frac{2\pi s}{A_m} - \frac{2\pi t}{B_n} + \alpha_m - \beta_n$

$$u_{st}^{p} = 2\pi k^{p}\left(r_n^{2} + d_m^{2} - 2r_n d_m \cos\varphi_{st}\right)^{\frac{1}{2}}.$$

Finally we obtain from (2.12) the formula for the crosspower spectrum C_{nj} between the horizontal ring averages h_n and h_j:

$$C_{nj} = \frac{1}{B_n B_j} \sum_{p=1}^{P} \sum_{s=1}^{B_n} \sum_{t=1}^{B_j} \phi^{p} \cdot \left\{ \tfrac{1}{2}\cos\varphi_{st}\, J_0\left(u_{st}^{p}\right) + \left(2\pi k^{p}\right)^{2} \left[2r_n r_j - \left(r_n^{2} + r_j^{2}\right)\cos\varphi_{st}\right] \cdot \left[\frac{1}{16} J_0\left(u_{st}^{p}\right) + \frac{1}{12} J_2\left(u_{st}^{p}\right) + \frac{1}{48} J_4\left(u_{st}^{p}\right)\right]\right\} \tag{2.16}$$

(<u>multimode crosspower between two horizontal ring averages</u>)

where $\varphi_{st} = \frac{2\pi s}{B_n} - \frac{2\pi t}{B_j} + \beta_n - \beta_j$

and $u_{st}^{p} = 2\pi k^{p}\left(r_n^{2} + r_j^{2} - 2 r_n r_j \cos\varphi_{st}\right)^{\frac{1}{2}}$

D. MULTICHANNEL INTERPOLATION FILTERS AND INTERPOLATION ERROR

For each $m = 1, 2, \ldots, M$, we wish to design a set of optimum interpolation filters G_{1m}, G_{2m}, ..., G_{Nm} to be used in estimating v_m from h_1, h_2, ..., h_N in multimode noise (i.e., G_{jm} is the

frequency filter to be applied to h_j to estimate v_m, $j=1, 2, \ldots, N$).

According to (1.4), the optimum interpolation filters G_{jm} are obtained by solving the matrix equation

$$\left(C_{nj}\right)\left(G_{jm}\right) = \left(S_{nm}\right) \tag{2.17}$$

where C_{nj} and S_{nm} are given by (2.15) and (2.16). It is interesting to note here that if all of the modal horizontal-vertical transfer functions K^p all are purely imaginary at all frequencies (which is the case for the usual theoretical model of a layered half-space [Laster-Linville, 1966]) then by (2.15) and (2.16), (C_{nj}) and (S_{nm}) are real matrices, and hence the filters G_{jm} are real also.

Once the optimum interpolation filters G_{jm} have been computed, the optimum interpolation error for v_m is given by (1.5) as

$$(\text{I. E.})_m = 1 - \left(1/V_m\right) \sum_{j=1}^{N} G_{jm} S_{jm}^{*} \tag{2.18}$$

SECTION III

ALTERNATIVE CROSSPOWER FORMULAS

If we assume a certain noise field and specify an array geometry, then by use of formulas (2.14-2.16) we may calculate the array crosspower spectra on an electronic computer to any desired degree of precision; these results may in turn be used to calculate the interpolation filters and interpolation error (formulas 2.17, 2.18). However, although formulas (2.14 - 2.16) are suitable for numerical calculation, they are exceedingly inelegant, and because of their complexity they can not give insight into how the array "works." How many rings should we have in our array? How does the response of a ring change if we double or triple the number of seismometers in the ring? What is the effect of varying the rotation angles α_m and β_n of the vertical and horizontal rings respectively? For a theoretical, i.e., non-computational, study of questions such as these, formulas (2.14-2.16) are no help. It is of interest, therefore, to derive alternative expressions for the array crosspower spectra which do give insight into the functioning of the array.

A. PRELIMINARIES

The principal tool in the derivation of our alternative crosspower expressions will be Neumann's formula (Watson, p. 358):

$$J_0\left(\left[x^2+y^2-2xy\cos\varphi\right]^{\frac{1}{2}}\right) = \sum_{q=0}^{\infty} \epsilon_q\, J_q(x)\, J_q(y)\cos q\varphi \tag{3.1}$$

where

$$\epsilon_q = \begin{cases} 1, & q = 0 \\ 2, & q > 0 \end{cases}$$

The series on the right is absolutely convergent for all real x, y, and φ. The convergence is uniform in x and y if x and y are restricted to lie in any finite interval. Hence one may differentiate the series term by term with respect x or y.

We shall also need two definitions from elementary number theory. (1) An integer q is a multiple of the integer N if and only if there exists an integer m such than mN = q. Thus, for example, zero is a multiple of all integers. 6 is a multiple of 2, 7 is not a multiple of 3. (2) Let N_1 and N_2 be positive integers. The least common multiple of N_1 and N_2 is the smallest positive integer N such that N is a multiple of both N_1 and N_2. Thus, the least common multiple of 3 and 7 is 21, the least common multiple of 6 and 8 is 24, and the least common multiple of 5 and 10 is 10.

Finally, we need to know the following fact: let q and N be integers, with $N \geq 1$, and let δ be any real number. Then

$$\sum_{n=1}^{N} \cos\left(q\left[\frac{2\pi n}{N} + \delta\right]\right) \tag{3.2}$$

$$= \begin{cases} N \cos q\,\delta, & \text{if } q \text{ is a multiple of } N \\ 0, & \text{if } q \text{ is not a multiple of } N \end{cases}$$

$$= \begin{cases} N \cos q\,\delta, & \text{if } q = 0, \pm N, \pm 2N, \pm 3N, \ldots \\ 0, & \text{if otherwise.} \end{cases}$$

B. INFINITE SERIES FORMULAS FOR CROSSPOWER SPECTRA

Let v_1 and v_2 be the averages of two concentric rings of regularly spaced vertical seismometers; in the j^{th} ring, let N_j = number of seismometers, r_j = radius, α_j = rotation angle, j = 1, 2. Then in a single mode noise field (Φ, K, k), the crosspower spectrum V_{12} between v_1 and v_2 is (2.10)

$$V_{12} = \frac{|K|^2 \Phi}{N_1 N_2} \sum_{m=1}^{N_1} \sum_{n=1}^{N_2} J_0\left(\left[u_1^2 + u_2^2 - 2u_1 u_2 \cos \varphi_{mn}\right]^{\frac{1}{2}}\right), \tag{3.3}$$

where

$$u_1 = 2\pi k r_1, \; u_2 = 2\pi k r_2, \; \varphi_{mn} = \frac{2\pi m}{N_1} - \frac{2\pi n}{N_2} + \alpha_1 - \alpha_2.$$

Hence, according to Neumann's formula (3.1) we have

$$V_{12} = \frac{|K|^2 \Phi}{N_1 N_2} \cdot \sum_{m=1}^{N_1} \sum_{n=1}^{N_2} \sum_{q=0}^{\infty} \epsilon_q J_q(u_1) J_q(u_2) \cos q \varphi_{mn}, \text{ or,}$$

reversing the order of summation,

$$V_{1,2} = \frac{|K|^2 \Phi}{N_1 N_2} \sum_{q=0}^{\infty} \left\{ \epsilon_q J_q(u_1) J_q(u_2) \cdot \sum_{m=1}^{N_1} \sum_{n=1}^{N_2} \cos q \varphi_{mn} \right\} \tag{3.4}$$

The inner double summation may be simplified with the aid of (3.2). Summing first on n, we see that

$$\sum_{m=1}^{N_1} \sum_{n=1}^{N_2} \cos q \varphi_{mn} = \sum_{m=1}^{N_1} \sum_{n=1}^{N_2} \cos\left(q\left[\frac{2\pi m}{N_1} - \frac{2\pi n}{N_2} + \alpha_1 - \alpha_2\right]\right)$$

is equal to 0 unless q is a multiple of N_2, by (3.2). On the other hand, if we reverse the order of summation we see that the same sum is equal to 0 unless q is also a multiple of N_1. Thus we see that the inner double summation in (3.4) is 0 unless q is a multiple both of N_1 and N_2, i.e., unless q is a multiple of N, where N = the least common multiple of N_1 and N_2. So, suppose that q <u>is</u> a multiple of N. Then, using (3.2) twice, we get

$$\sum_{m=1}^{N_1} \sum_{n=1}^{N_2} \cos\left(q\left[\frac{2\pi m}{N_1} - \frac{2\pi m}{N_2} + \alpha_1 - \alpha_2\right]\right) \tag{3.5}$$

$$= \sum_{m=1}^{N_1} N_2 \cos\left(q\left[\frac{2\pi m}{N_1} + \alpha_1 - \alpha_2\right]\right)$$

$$= N_1 N_2 \cos\left(q\left[\alpha_1 - \alpha_2\right]\right)$$

Hence, (3.4) becomes

$$V_{1,2} = |K|^2 \Phi \sum_{q=0}^{\infty} \epsilon_{qN} J_{qN}(u_1) J_{qN}(u_2) \cos\left(qN[\alpha_1 - \alpha_2]\right), \quad \text{or}$$

$$V_{1,2} = |K|^2 \Phi J_0(u_1) J_0(u_2) + 2|K|^2 \Phi \sum_{q=1}^{\infty} J_{qN}(u_1) J_{qN}(u_2) \cos\left(qN[\alpha_1 - \alpha_2]\right) \tag{3.6}$$

(single-mode crosspower spectrum between two vertical ring averages)

where

$u_j = 2\pi k r_j$

r_j = radius,

α_j = rotation angle for the j^{th} ring

N = least common multiple of N_1 and N_2

$j = 1, 2$

Now, let h_1 and h_2 be the averages of two concentric rings of radially-oriented horizontals, where u_j, r_j, α_j, N_j, and N are the same as above. We wish to derive expressions for $S_{1,2}$ = crosspower between h_1 and v_2, and $C_{1,2}$ = crosspower between h_1 and h_2. In view of equations (2.2), (2.3), and (3.1), we see that $S_{1,2}$ and $C_{1,2}$ can be obtained from (3.6) as follows:

$$S_{1,2} = -iK\Phi \frac{\partial}{\partial u_1}\left(\sum_{q=0}^{\infty} \epsilon_{qN} J_{qN}(u_1) J_{qN}(u_2) \cos\left[qN(\alpha_1 - \alpha_2)\right]\right)$$

$$C_{1,2} = \Phi \cdot \frac{\partial^2}{\partial u_2 \partial u_1}\left(\sum_{q=0}^{\infty} \epsilon_{qN} J_{qN}(u_1) J_{qN}(u_2) \cos\left[qN(\alpha_1 - \alpha_2)\right]\right)$$

We may carry out the indicated differentiations term by term, to obtain

$$S_{1,2} = iK\Phi J_1(u_1) J_0(u_2) - 2iK\Phi \sum_{q=1}^{\infty} J'_{qN}(u_1) J_{qN}(u_2) \cos\left(qN\left[\alpha_1 - \alpha_2\right]\right) \tag{3.7}$$

(single-mode crosspower spectrum between horizontal and vertical ring averages)

and

$$C_{1,2} = \Phi J_1(u_1) J_1(u_2) + 2\Phi \sum_{q=1}^{\infty} J'_{qN}(u_1) J'_{qN}(u_2) \cos\left(qN\left[\alpha_1 - \alpha_2\right]\right) \tag{3.8}$$

(single-mode crosspower spectrum between two horizontal ring averages).

We could rewrite (3.7) and (3.8) by substituting the identity

$$J'_{qN}(u_j) = \tfrac{1}{2}\left(J_{qN-1}[u_j] - J_{qN+1}[u_j]\right).$$

Let us express formulas (3.6-3.8) in the form

$$V_{1,2} = |K|^2 \Phi \cdot J_0(u_1) J_0(u_2) + e_1(N, u_1, u_2) \tag{3.9}$$

$$S_{1,2} = iK\Phi J_1(u_1) J_0(u_2) + e_2(N, u_1, u_2) \quad \text{and}$$

$$C_{1,2} = \Phi J_1(u_1) J_1(u_2) + e_3(N, u_1, u_2)$$

where

$$e_1(N, u_1, u_2) = 2|K|^2 \Phi \sum_{q=1}^{\infty} J_{qN}(u_1) J_{qN}(u_2) \cos\left[qN(\alpha_1 - \alpha_2)\right], \text{ etc.}$$

The fact that (3.1) is absolutely convergent implies that for any fixed values of u_1 and u_2 we have

$$\lim_{N \to \infty} e_j(N, u_1, u_2) = 0, \; j = 1, 2, 3. \tag{3.10}$$

Since N is the least common multiple of N_1 and N_2, the numbers of seismometers in our two rings, (3.10) shows that if we hold the radii of our rings constant and let N_1 and N_2 increase to infinity, then the crosspower spectra (3.9) approach the limiting values

$$\begin{aligned} V_{1,2} &= |K|^2 \Phi \cdot J_0(u_1) J_0(u_2) \\ S_{1,2} &= iK\Phi J_1(u_1) J_0(u_2) \\ C_{1,2} &= \Phi J_1(u_1) J_1(u_2) \end{aligned} \tag{3.11}$$

That is, (3.11) gives the crosspower spectra that would be obtained by using "circular seismometers" whose outputs are respectively the average integrated motion of all points in continuous rings with respective radii r_1 and r_2. It is unfortunate that such instruments do not exist, because they would be the ideal instruments to employ in horizontal-vertical interpolation processing.

In fact, perfect interpolation would be possible if circular seismometers could be used. For, suppose our isotropic noise field contains P non-zero modes with modal parameters Φ_p, K_p, and k_p, $p = 1, 2, \ldots, P$, (see section 2C), and our array consists of one circular

vertical seismometer v with radius d, and P circular horizontal seismometers h_n with radii r_n, $n = 1, 2, \ldots, P$. We shall show that h_1, $h_2, \ldots,$ and h_p can be used to predict v with zero interpolation error. Following our usual notation, let V = autopower spectrum of v, S_n = crosspower spectrum between h_n and v, and C_{nj} = crosspower spectrum between h_n and h_j. From (3.11) we find

(3.12)

$$V = \sum_{p=1}^{P} |K_p|^2 \Phi_p J_0 (2\pi k_p d)^2$$

$$S_n = \sum_{p=1}^{P} iK_p \Phi_p J_1 (2\pi k_p r_n) J_0 (2\pi k_p d)$$

$$C_{nj} = \sum_{p=1}^{P} \Phi_p J_1 (2\pi k_p r_n) J_1 (2\pi k_p r_j)$$

Define the matrices Φ and $A = (A_{pn})$ by

$$\Phi = \mathrm{diag}\,(\Phi_1, \Phi_2, \ldots, \Phi_P) \text{ and}$$

$$A_{pn} = J_1 (2\pi k_p r_n),$$

and define the column vector K by

$$K = \left(K_p J_0 (2\pi k_p d)\right)$$

Then the matrix $C = (C_{nj})$, the column vector $S = (S_n)$, and the 1×1 matrix V satisfy

(3.13)

$$C = A^t \Phi A$$

$$S = iA^t \Phi K$$

$$V = (K^*)^t \Phi K.$$

Now, the optimum interpolation filters $G_1, \ldots, G_p$ are found by solving the equation $CG = S$ (eq. 1.4) for G, where G is the column vector (G_p). But by virtue of (3.13) this equation becomes

$$A^t \Phi AG = iA^t \Phi K \tag{3.14}$$

Assuming the matrix A to be nonsingular, we can solve the equation $AG = iK$ to obtain

$$G = iA^{-1}K \tag{3.15}$$

and this solution is automatically a solution of (3.14). (Note that this solution is independent of Φ, i.e., the optimum interpolation filters do not depend upon the autopower spectra of the various modes for the case of circular seismometers.) The interpolation error is given by

$$I.E. = 1 - \frac{1}{T} \sum_{j=1}^{P} G_j \, S_j^* \qquad \text{(eq. 1.5)} \tag{3.16}$$

Now, regarding 1 x 1 matrices as numbers, we note that

$$\sum_{j=1}^{P} G_j S_j^* = (S^*)^t G$$

Hence

$$P.E. = 1 - (1/V)([S^*]^t G)$$

$$= 1 - (1/V)\,(-i\,[K^*]^t\,\Phi\,A \cdot iA^{-1}K), \qquad \text{by (3.13 and 3.15)}$$

$$= 1 - \frac{(K^*)^t\,\Phi K}{(K^*)^t\,\Phi K} = 0$$

Thus, in a noise field with P modes, it is possible to achieve perfect estimation of a circular vertical seismometer by employing P circular horizontal seismometers.

On the other hand, a close look at the equations of the preceding paragraph will show that if we had tried to estimate v by using fewer than P circular horizontal seismometers, we should have obtained rather poor results in general. That is, one needs at least P circular horizontals to handle P modes of noise. It is clear that this same rule of thumb must apply in the case of the finite arrays which are the object of this study, since a circular seismometer is the ideal limiting case of a ring containing finitely many seismometers. Thus, the most obvious but perhaps the most important design criterion for horizontal-vertical interpolation arrays is

Number of horizontal rings $\geq$ number of noise modes. (3.17)

We have seen (eq. 3.10) that as we let the number of seismometers in each ring approach infinity, the response of an array of rings of seismometers approximates the response of an array of circular

seismometers. But finite rings approximate circular seismometers in another sense as well, for it may be shown that

$$\lim_{u_1, u_2 \to 0} \frac{e_1(N, u_1, u_2)}{|K|^2 \Phi J_0(u_1) J_0(u_2)} = 0, \quad \text{for } N \geq 1 \tag{3.18}$$

$$\lim_{u_1, u_2 \to 0} \frac{e_2(N, u_1, u_2)}{iK \Phi J_1(u_1) J_0(u_2)} = 0, \quad \text{for } N > 2$$

and

$$\lim_{u_1, u_2 \to 0} \frac{e_3(N, u_1, u_2)}{\Phi J_1(u_1) J_1(u_2)} = 0, \text{ for } N > 2.$$

These equations together with (3.9) show that the crosspower spectra for finite rings approximate the crosspower spectra for circular seismometers if the $u_j = 2\pi k r_j$ are small (and the number of seismometers in each horizontal ring is greater than 2). Thus, suppose we have a fixed finite ring array with each horizontal ring containing at least 3 seismometers and suppose that the number of horizontal rings is greater than or equal to the number of noise modes present. Then the optimum interpolation error approaches zero as the maximum wavenumber

$$\max_{1 \le p \le P} \left(k_p(f) \right)$$

approaches zero. Conversely it can be shown that the optimum interpolation error approaches 1 as

$$\min_{1 \le p \le P} \left(k_p(f) \right)$$

gets very large.

We shall now discuss some further consequences of equations (3.6) - (3.8). For convenience, let

$$(N_1, r_1, \alpha_1) x (N_2, r_2, \alpha_2)$$

denote the crosspower between two rings, where for the j^{th} ring,

N_j = number of seismometers

r_j = radius

α_j = rotation angle

j = 1, 2

Then (3.6) - (3.8) show that

$$(N_1, r_1, \alpha_1) \times (N_2, r_2, \alpha_2) = (N, r_1, \alpha_1) \times (N, r_2, \alpha_2) = (N_2, r_1, \alpha_1) \times (N_1, r_2, \alpha_2) \tag{3.19}$$

where

$$N = \text{least common multiple of } N_1 \text{ and } N_2$$

Thus, for example, the crosspower spectrum between a 6-element ring and a 7-element ring is the same as the crosspower spectrum between two 42-element rings. Since a 42-element ring is a closer approximation to the ideal case of a circular seismometer than is either a 6-element ring or a 7-element ring, we have evidence that is very suggestive of a possible advantage to not having the same number of seismometers in all rings of the array. There may also be an advantage in varying the rotation angles α_j from ring to ring. For example, in the case of 6- and 7-element horizontal rings, we have

(a)

$$(6, r_1, 0) \times (7, r_2, 0) = \Phi \cdot J_1(u_1) J_1(u_2)$$

$$+ \quad 2\,\Phi \cdot \left\{ J'_{42}(u_1) J'_{42}(u_2) + J'_{84}(u_1) J'_{84}(u_2) + \dots \right\}$$

whereas

(b)

$$(6, r_1, 0) \times (7, r_2, \tfrac{\pi}{84}) = \Phi \cdot J_1(u_1) J_1(u_2)$$

$$+ 2\,\Phi \left\{ - J'_{84}(u_1) J'_{84}(u_2) + J'_{168}(u_1) J'_{168}(u_2) - \dots \right\}$$

In (b) we have obtained a much better approximation to the crosspower between two circular seismometers than in (a).

However, the question of exactly how much advantage, if any, there is to varying the number of seismometers and rotation angles from ring to ring is very difficult to attack from a purely theoretical point of view. An equally difficult problem is that of optimizing the choice of the ring radii once the number of rings has been decided upon. Unfortunately it seems at the present time that the best way of investigating these problems is to make numerical calculations of interpolation error for various assumed noise models and array geometries, and arrive at array geometries suitable for a given noise model through trial and error. Section V gives an account of the first steps that have been taken in such a program.

SECTION IV

ARRAY RESPONSE FOR A GIVEN SET OF INTERPOLATION FILTERS

A. INTERPOLATION ERROR FOR AN ARBITRARY NOISE MODE

Until now we have been concerned with the problem of designing filters to minimize the interpolation error for a given multimode noise field. Let us now take a slightly different point of view. Suppose we have an array consisting of a ring v of verticals and N rings h_1, $h_2, \ldots, h_N$ of horizontals, and suppose we have already designed a set of filters G_1, $G_2, \ldots, G_N$. What is the response of the given array (with the given set of filters) to an arbitrary noise mode with modal parameters Φ, K, and k? While the filters G_n may be optimum interpolation filters for some particular multimode noise field, we ask how well the array will perform in noise for which the filters may not be optimum.

For the noise mode (Φ, K, k) the output e(t) of the interpolation processor is given by (1.1). As before, let E = autopower of e, V = autopower of v, S_n = crosspower between h_n and v, and C_{nj} = crosspower between h_n and h_j. We have (1.3),

$$E = V - \sum_{n=1}^{N} G_n^* S_n - \sum_{n=1}^{N} G_n S_n^* + \sum_{n=1}^{N} \sum_{j=1}^{N} G_n^* C_{nj} G_j$$

Now, in the single-mode crosspower formulas (2.10-2.12), note that the only places where Φ and K occur are outside the summations as multiplicative factors.

We may therefore write

$$V = |K|^2 \Phi \cdot a(f, k), \tag{4.1}$$

$$\sum_{n=1}^{N} G_n^* S_n = iK\Phi \cdot b(f, k), \text{ and}$$

$$\sum_{n=1}^{N} \sum_{j=1}^{N} G_n^* C_{nj} G_j = \Phi \cdot c(f, k)$$

where the functions a, b, and c are independent of Φ and K. (a, b, and c do depend upon the array geometry and upon the filters G_n, but these are regarded as fixed.) It may be shown that a and c are real and non-negative, and $a \leq 1$; b may be complex, but b is real if the filters G_n are.

Substituting (4.1) into (1.3), we find

$$E = \Phi \cdot \left[a \cdot |K|^2 - ibK + ib^* K^* + c \right] \tag{4.2}$$

$$= a \cdot |K|^2 \Phi \cdot \left\{ \left| 1 - \frac{b^*}{a} \cdot \frac{1}{ik} \right|^2 + \frac{1}{|K|^2} \left(\frac{c}{a} - \frac{|b|^2}{a^2} \right) \right\}$$

Hence the interpolation error I. E. = E/V is

$$\text{I. E.} = \left| 1 - \frac{b^*}{a} \cdot \frac{1}{iK} \right|^2 + \frac{1}{|K|^2} \left(\frac{c}{a} - \frac{|b|^2}{a^2} \right)$$

Let us define the total error (T. E.) for our processor to be

$$T.E. = E/(\text{autopower of a single vertical seismometer})$$
$$= E/|K|^2 \Phi = a \cdot (I.E.)$$

Then, by (4.3),

$$T.E. = a \cdot \left| 1 - \frac{b^*}{a} \cdot \frac{1}{iK} \right|^2 + \frac{1}{|K|^2} \cdot \left(c - \frac{|b|^2}{a} \right) \tag{4.4}$$

The total error T. E. is a better measure than I. E. of how well our processor is succeeding in removing surface-wave noise from a vertical P-wave trace, since T. E. takes into account the fact that some noise reduction is effected by wavenumber aliasing between the seismometers in the vertical ring v. If the vertical ring has radius zero and contains only one seismometer, then T. E. = I. E.

B. GRAPHICAL REPRESENTATION OF ARRAY RESPONSE

In the study of vertical arrays it has been found useful to represent array response graphically by contouring the power response of the array in the k_x-k_y plane for selected fixed frequencies (see, for example, Burg, 1964, pp. 710-712).* A similar method for graphical representation of the response of horizontal-vertical interpolation arrays will be described in this subsection.

Now, the response of a vertical array to a wave propagating across the surface depends upon frequency, wavenumber, and direction. However, the response of a horizontal-vertical interpolation array depends upon yet another variable: the horizontal-vertical transfer function K.

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*The term "vertical array" is used here to describe a planar array with only vertical-component seismometers.

It is convenient to simplify the problem by eliminating at least one of the four variables; as we have done throughout this report, we will eliminate the parameter of direction by assuming isotropy. Probably not much is lost by this simplification since, for wavenumbers small enough to make good interpolation feasible, directional effects are probably negligible (see Section III).

In the usual situation the filters G_n will be real (see Section IID), and hence the functions $a(f, k)$, $b(f, k)$, and $c(f, k)$ will be real also; and for our isotropic single-mode Rayleigh wave noise field (Φ, K, k), K will be purely imaginary. If we let $H = 1/iK$, then H is real, and equations (4.3) and (4.4) simplify to

$$\text{I.E.} = 1 - \frac{2b}{a} H + \frac{c}{a} H^2 \qquad \text{and} \tag{4.5}$$

$$\text{T.E.} = a - 2bH + cH^2$$

That is, if we hold f and k fixed and let H vary from $-\infty$ to $+\infty$, I.E. describes a parabola with a minimum value of $1 - \frac{b^2}{ac}$, attained when $H = \frac{b}{c}$. If $H = 0$ or $H = \frac{2b}{c}$, then I.E. = 1. Similarly, T.E. describes a parabola with a minimum value of $a - \frac{b^2}{c}$, attained when $H = b/c$. If $H = \frac{b}{c} \pm \frac{1}{c} \sqrt{b^2 - ac + c}$, then T.E. = 1.

Now, select a set of frequencies f_j, $j = 1, 2, \ldots$. For each fixed frequency f_j, plot the curves (1) $H = 0$, (2) $H = b/c$, (3) $H = \frac{2b}{c}$, (4) $H = \frac{b}{c} + \frac{1}{c} \sqrt{b^2 - ac + c}$, and (5) $H = \frac{b}{c} - \frac{1}{c} \sqrt{b^2 - ac + c}$ in the H-k plane (see Figure 4.1). For points (H, k) lying between curves (1) and (3), I.E. < 1; for all other points, I.E. ≥ 1. Similarly, for all points lying between curves (4) and (5), T.E. ≤ 1; for all other points, T.E. ≥ 1.

Two possible refinements to the above might be a) writing down the value of T.E. or I.E. at selected points along the curve (2) of minima, or b) contouring either T.E. or I.E. in the H-k plane, using either a linear or logarithmic scale for the contour intervals.

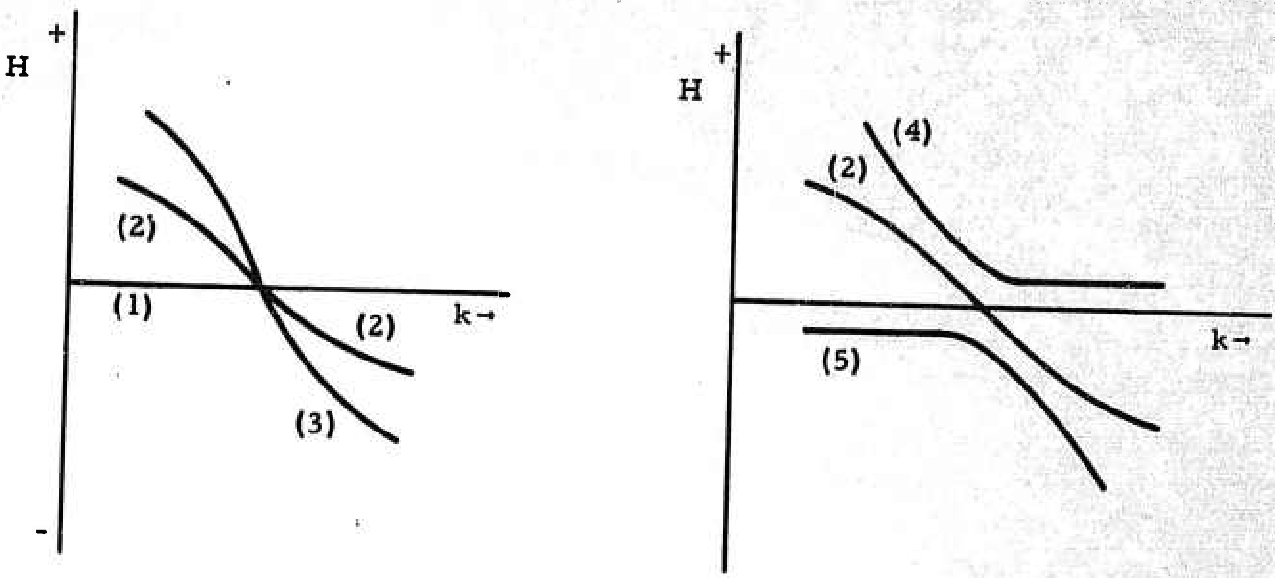

Figure 4.1. Graphical Representation of Array Response for Fixed Frequency f_j (Sketch)

C. RESPONSE OF ARRAY TO NON-VERTICAL P-WAVES

If an emergent P-wave is not vertical, then our interpolation processor will distort the signal to a certain extent. Exactly to what extent the detectability of P will be affected is related to the quantity E/Q, where E is the autopower of the output (due to the P-wave) of the interpolation processor, and Q is the autopower of the output (due to the P-wave) of a single vertical seismometer. E/Q is what we have previously called the <u>total error</u>, and we shall continue to call it by that name, although it is now somewhat misleading to do so since in the present context it is desirable for T.E. = E/Q to be large rather than small. For a vertical P-wave, T.E. = 1.

Suppose that we have P-waves whose apparent angle of emergence deviates from the vertical by an angle θ. As usual, assume that the energy is isotropic with respect to horizontal azimuth and uncorrelated from direction to direction. Let $k_0(f)$ be the inline wavenumber of the P-waves. Then there is an apparent horizontal wavenumber, which is approximately given by

$$k = k_0 \sin\theta \tag{4.7}$$

and an apparent horizontal-vertical transfer junction $K = \cot\theta$. Now assume that the filters G_n are real, so that $a(f, k)$, $b(f, k)$, and $c(f, k)$ are all real, where k is given by (4.7). Since K is also real, eq. (4.4) reduces to

$$\text{T.E.} = a + \frac{c}{K^2} = a + c \tan^2\theta \tag{4.8}$$

Note that if the vertical ring v has radius 0 and contains only seismometer, then a = 1 and hence 4.8 becomes

$$\text{T.E.} = 1 + c \tan^2\theta, \tag{4.9}$$

showing that in this case, the vertical amplitude of incident P-waves is always increased by using the interpolation processor.

For fixed frequencies, it would be possible to plot T.E. as a junction of θ and k_0, using (4.8). Although a detailed investigation has not been made, it appears that even for a multi-element vertical ring, small deviations from the vertical angle of emergence will not significantly affect P-wave signal amplitudes.

SECTION V

NUMERICAL CALCULATIONS

In order to check the theory presented in the preceding sections on concrete examples, a Fortran program for use on the IBM 7044 has been written which (1) accepts as inputs a) the geometrical parameters describing a specific horizontal-vertical prediction array and b) the modal parameters Φ^p, K^p, k^p for a specific assumed multi-mode surface wave noise field, (2) calculates V_m, S_{nm}, and C_{nj} according to (2.14-2.16), (3) calculates the optimum interpolation filters G_{jm} according to (2.17), and (4) calculates the optimum interpolation error and total error according to (2.20). Provision is made in the program for adding a given fraction ϵ of random noise to the predicting channels $h_1, \ldots, h_N$, as follows: we simply replace the matrix (C_{nj}) in (2.17) by the matrix (D_{nj}) defined by

$$(D_{nj}) = (C_{nj}) + \lambda I$$

where

$$\lambda = \epsilon \ . \ \frac{C_{11} + C_{22} + \ldots + C_{NN}}{N} \qquad \text{and}$$

I = the identity matrix. All computations have used the three values $\epsilon = 0$, $\epsilon = 0.01$, and $\epsilon = 0.10$ meaning, respectively, no uncorrelated noise, 1 percent uncorrelated noise, and 10 percent uncorrelated noise on the predicting channels.

In order to obtain fairly realistic examples of noise fields, we have used the first 3 theoretically computed noise modes for a 12 layer

theoretical UBO model (T.I. Array Research Semiannual Report No. 2, p. I-34). The horizontal and vertical amplitude functions U^p and V^p, $p = 1, 2, 3$ for the first three modes are shown in Figures (5.1 - 5.3). K^p in equal to V^p/iU^p, and Φ^p is equal to $(U^p)^2$. The dispersion curves for modes 1-3 are shown in Figure (5 4).

In Figure 5.2, notice that the horizontal amplitude for mode 2 vanishes at a frequency of about 0.5 cps. At this frequency, no estimation of a vertical component from horizontal components is possible for mode 2. Hence by eq. (4.5), the interpolation error equals 1 for a noise field consisting solely of mode 2. However, by properly choosing the radius d for a vertical ring v, we can make the <u>total error</u> small, as shown by eq. (4.6). For in eq. (4.6), T.E. = a(f, k) for H = O, and for the proper choice of d, a(f, k) can be made very small for the frequency 1.05 cps. The correct choice for d turns out to be approximately 0.892 km.

Four cases have been computed. The assumed isotropic noise fields for the four cases were:

CASE A: Modes 1 and 2. Frequency range: 0.05-2 cps

CASE B: Modes 1 and 2. Frequency range: 0.5-1.5 cps

CASE C: Modes 1 and 2. The horizontal and vertical amplitudes of mode 2 were multiplied by 7, so as to make the vertical amplitudes of the 1st and 2nd modes approximately equal at 1 cps. Frequency range : 0.05-2cps.

CASE D: Modes 1, 2, and 3. Horizontal and vertical amplitudes for mode 2 were again multiplied by 7, and horizontal and vertical amplitudes for mode 3 were multiplied by 10. Frequency range: 0.5-1.5 cps.

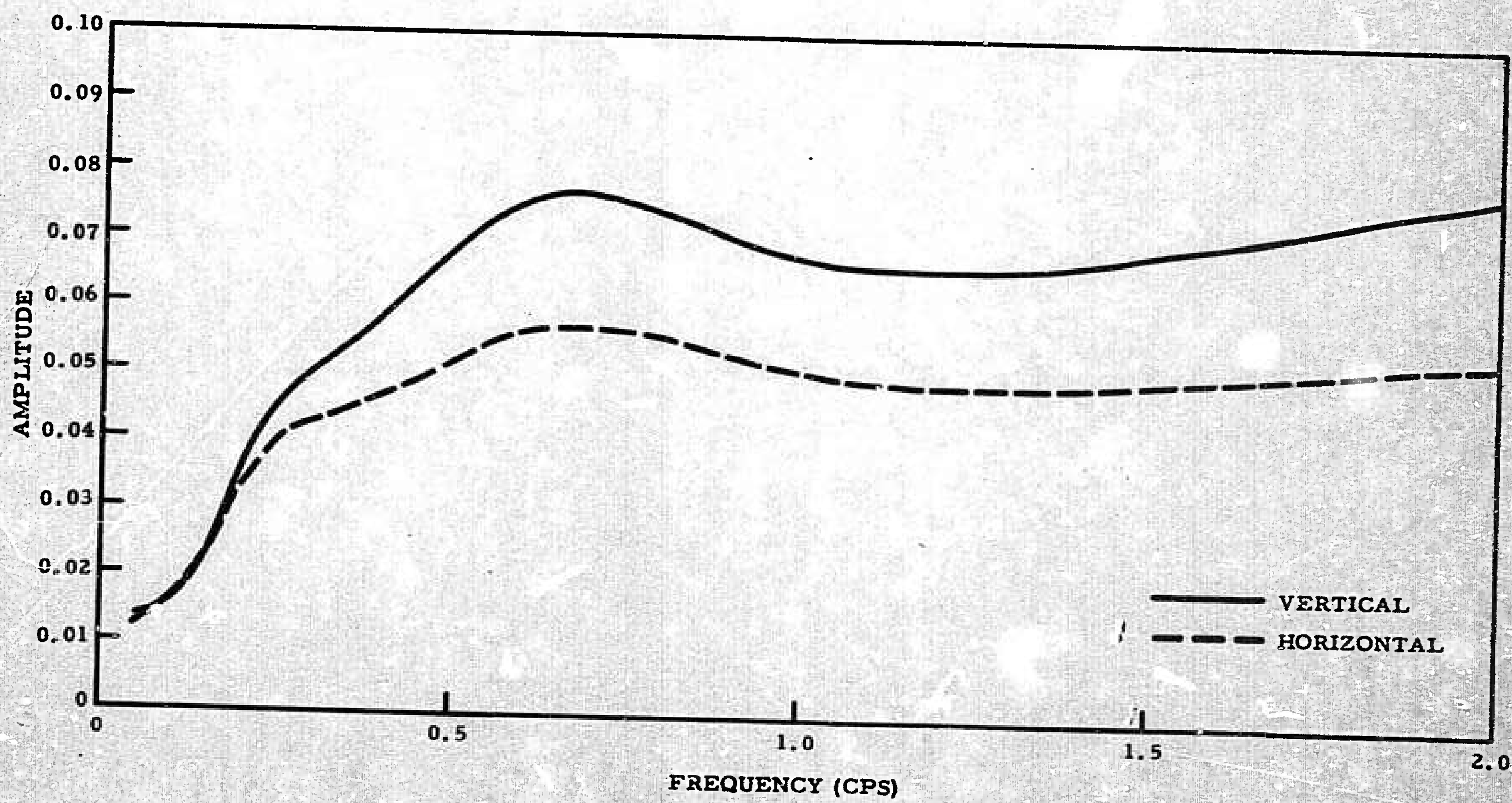

Figure 5.1. Horizontal and Vertical Amplitudes for Mode 1

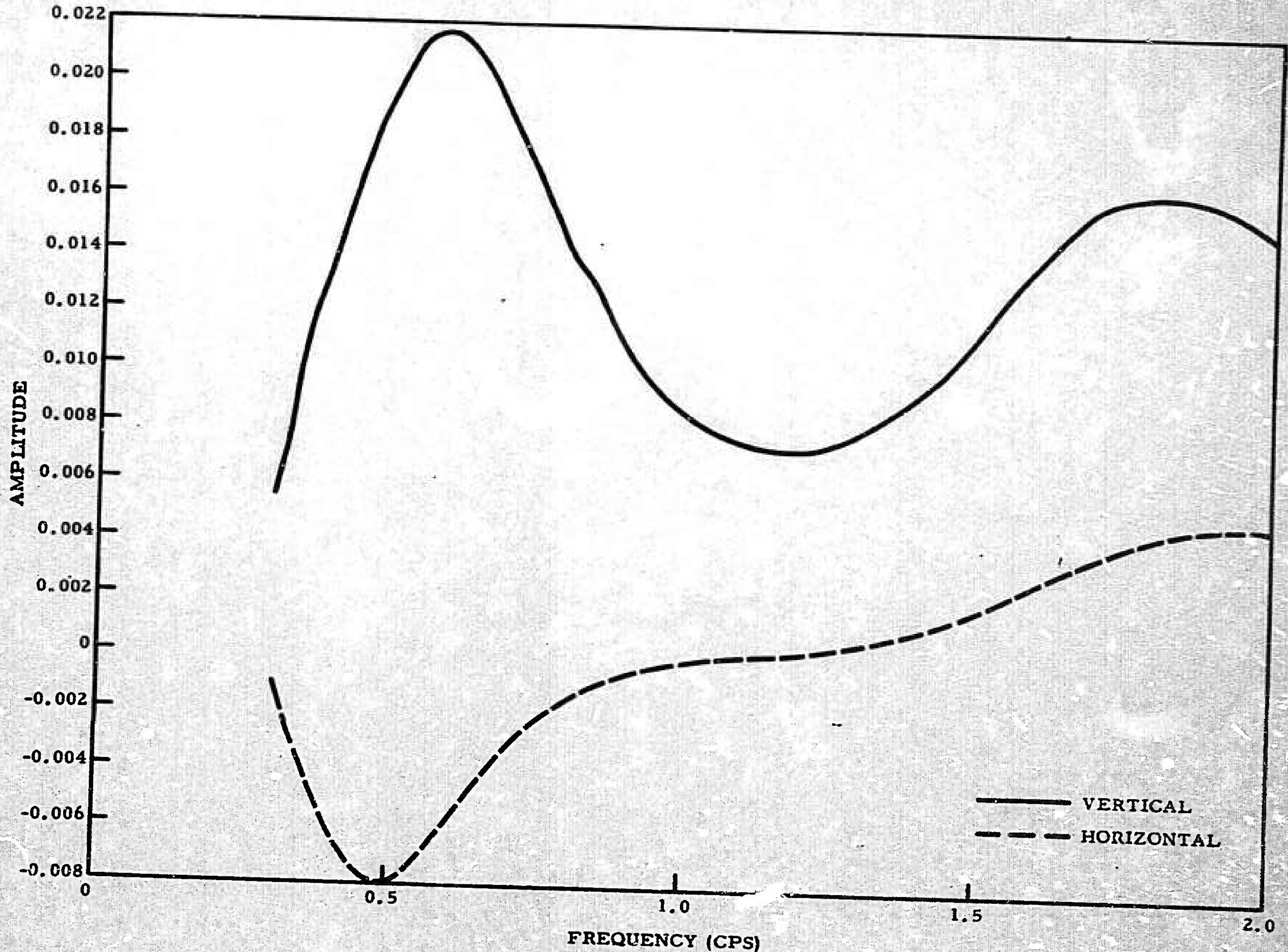

Figure 5.2. Horizontal and Vertical Amplitudes for Mode 2

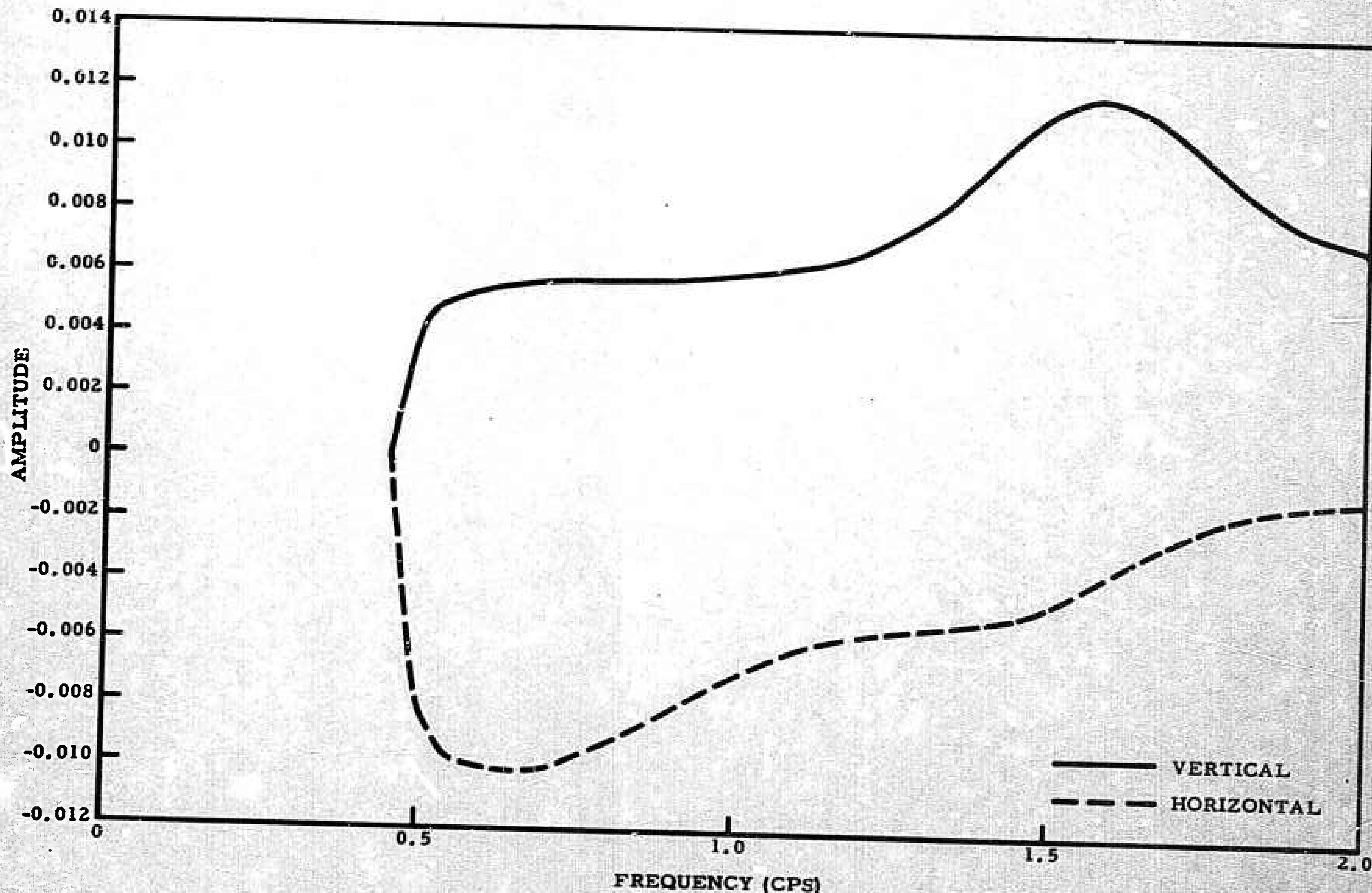

Figure 5.3. Horizontal and Vertical Amplitudes for Mode 3

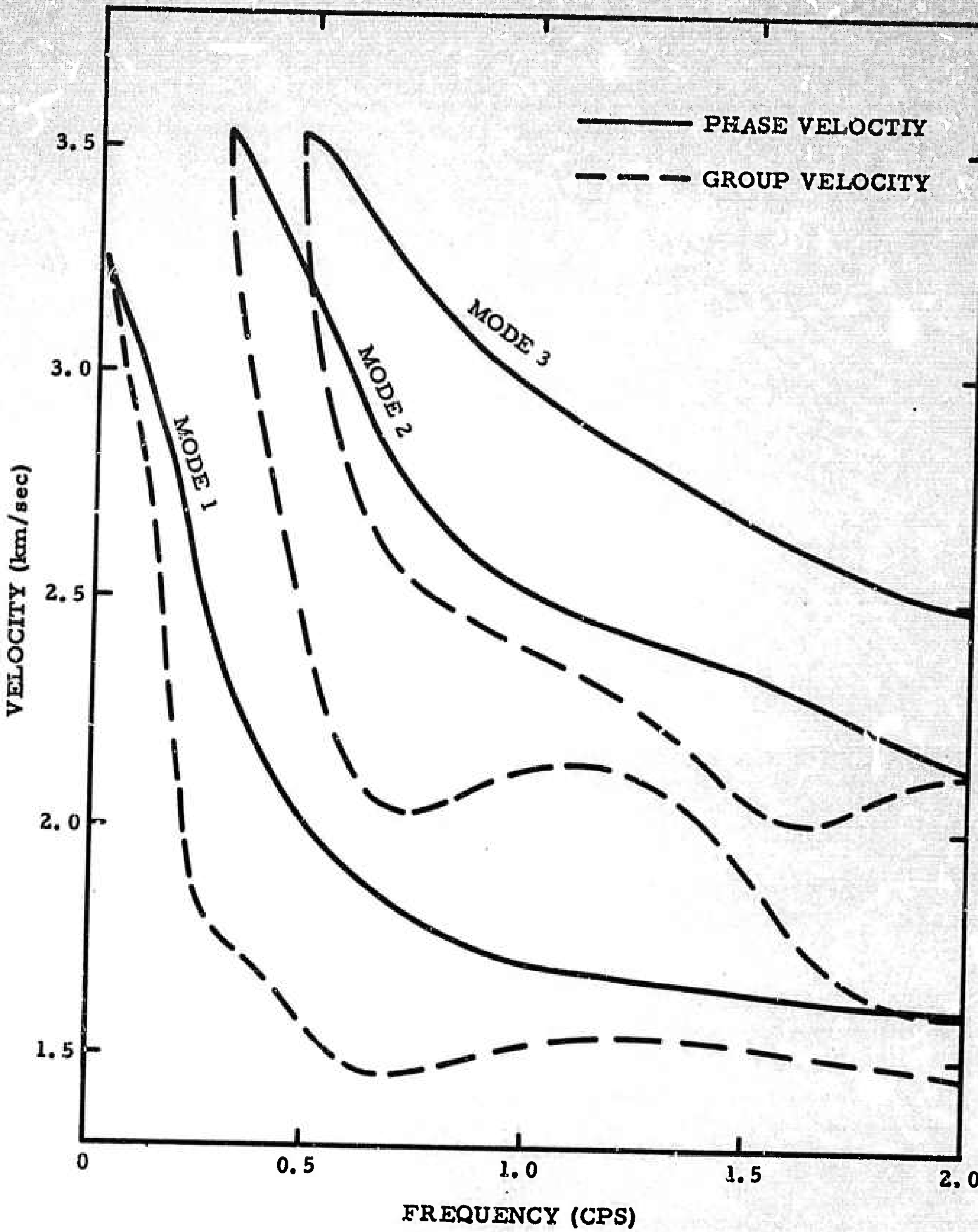

Figure 5.4. Dispersion Curves for Modes 1 through 3

The assumed array geometries for each case were:

CASE A: One vertical ring, radius = 0.892 km. Two horizontal rings, radii = 0.5 and 0.25 km. Six seismometers in each ring.

CASE B: One vertical ring, radius = 0.0, containing one seismometer. One horizontal ring, radius = 0.5 km, containing six seismometers.

CASE C: Same as Case A.

CASE D: One vertical ring, radius = 0.892 km. Three horizontal rings, radii = 0.5, 0.4, and 0.25 km. Six seismometers in each ring.

All rotation angles were zero for the cases considered.

In Figures 5.5 - 5.12, the computed interpolation filters, interpolation error, and total error are shown for these four cases, for an assumed 1 percent uncorrelated noise on the predicting channels.

In Cases A and B the first mode is much larger than the second mode, so that we almost have a single-mode noise field. However Figures 5.6 and 5.8 show that we still obtain much better performance by using 2 rings of horizontals and a ring of verticals than can be obtained by using a single ring of horizontals and a single central vertical.

In the true multimode cases C and D we have not obtained good performance for frequencies less than 0.8 cps, probably because for those frequencies the wavenumbers are small enough to make some of the horizontal ring crosspower spectra almost as small as our assumed random noise level. (By equation (3.9), the horizontal ring crosspower spectra approach zero as k approaches zero.) However, for frequencies greater than 0.8 cps, useful results are obtained. It may be that by varying the array geometrics one may broaden the band of frequencies over which good interpolation is possible. Further study is indicated.

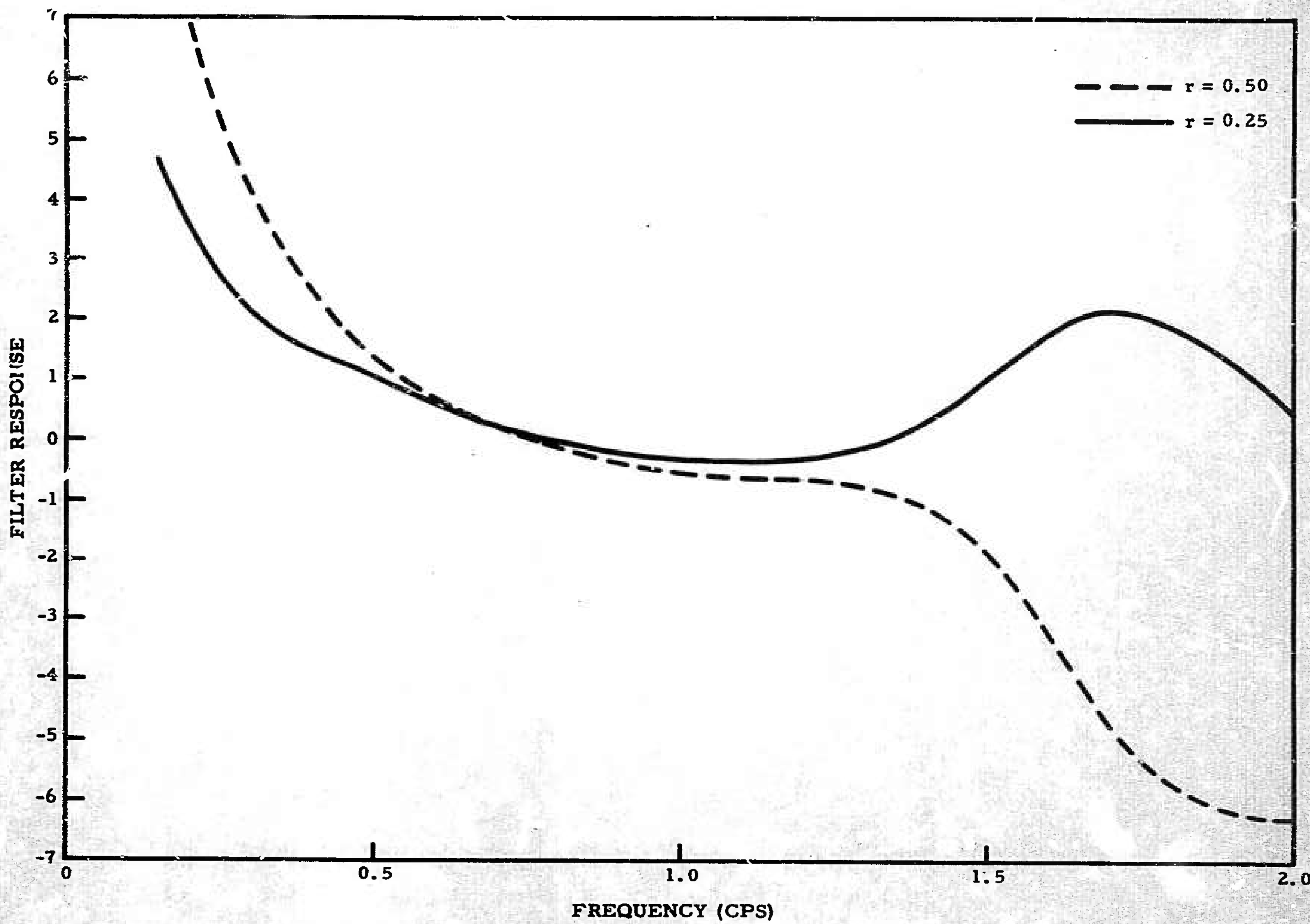

Figure 5.5. Computed Interpolation Filters for Case A (Modes 1 and 2, 1 Percent Uncorrelated Noise on Horizontal Rings)

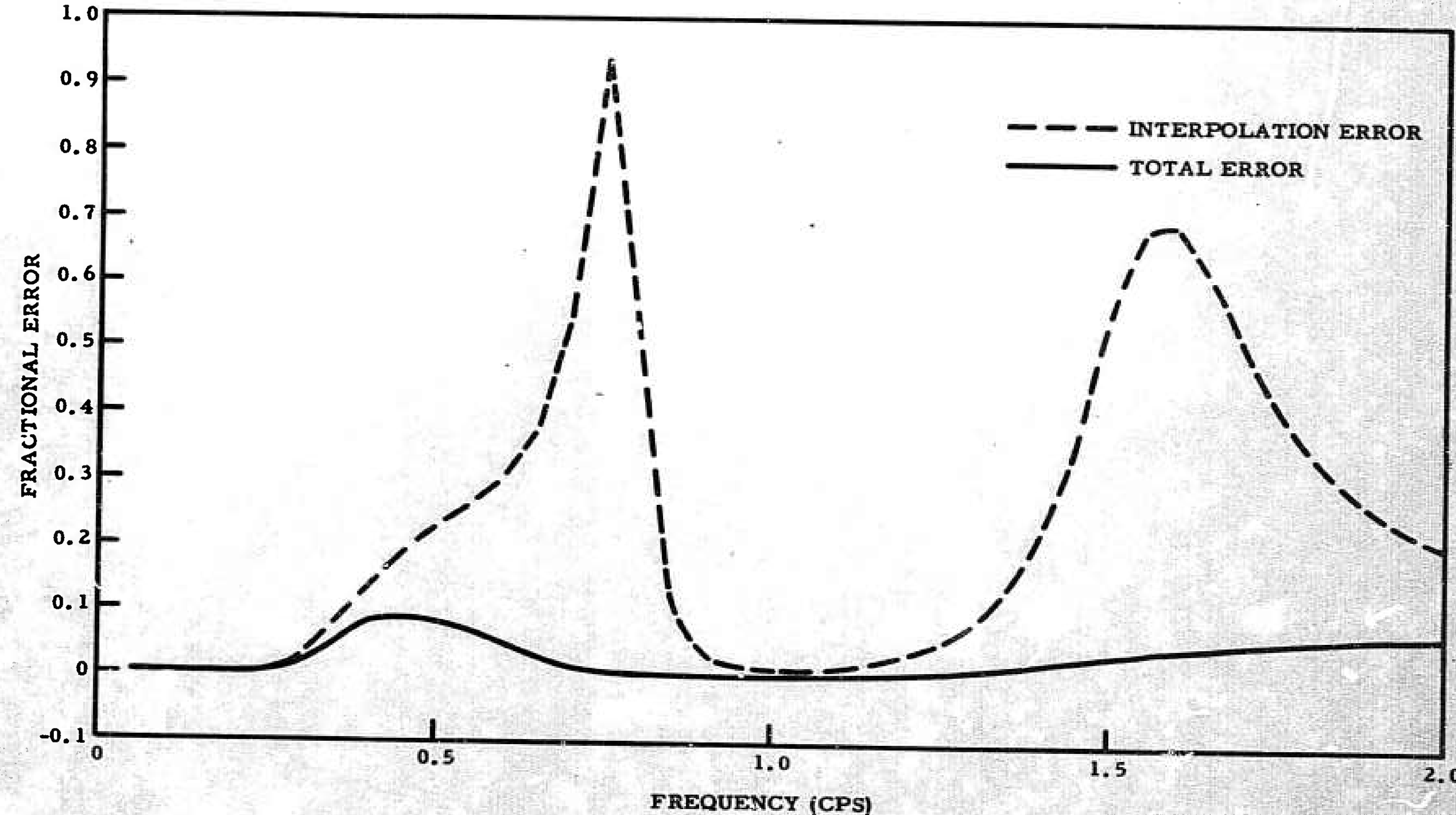

Figure 5.6. Interpolation Error and Total Error for Case A

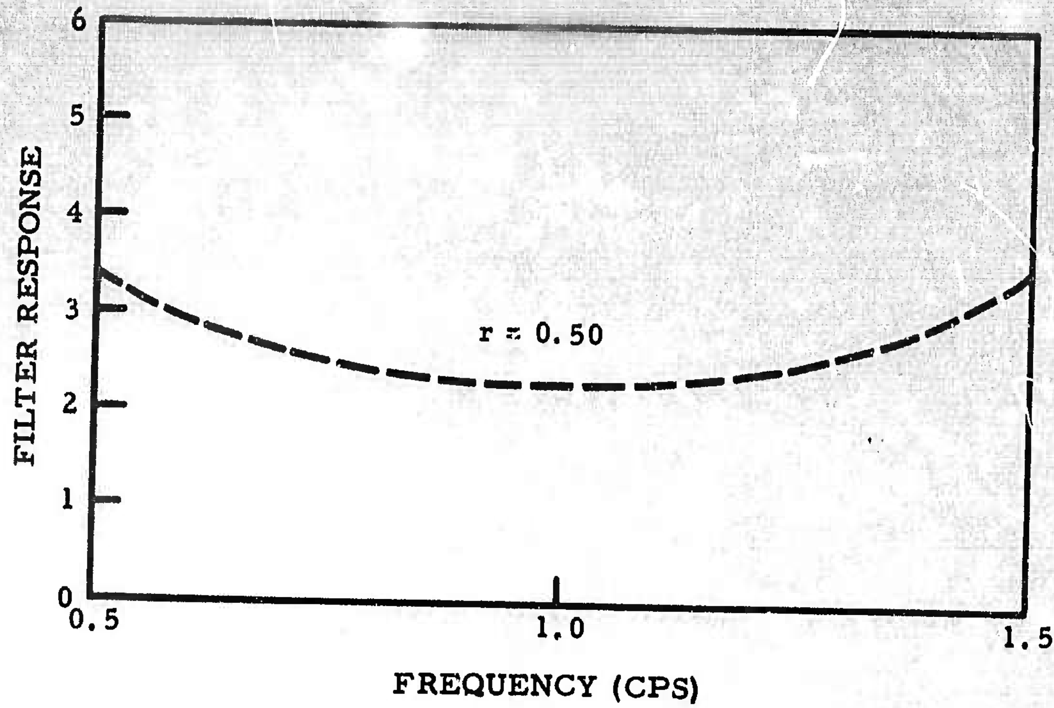

Figure 5.7. Computed Interpolation Filter for Case B (Modes 1 and 2, 1 Percent Uncorrelated Noise on Horizontal Ring)

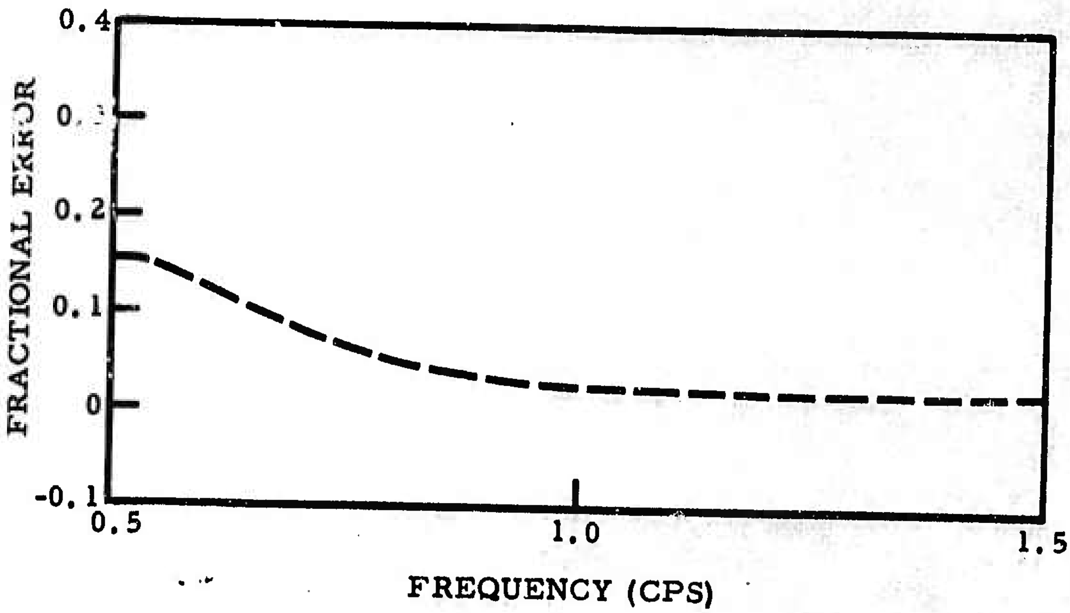

Figure 5.8. Interpolation Error (Equals Total Error) for Case B

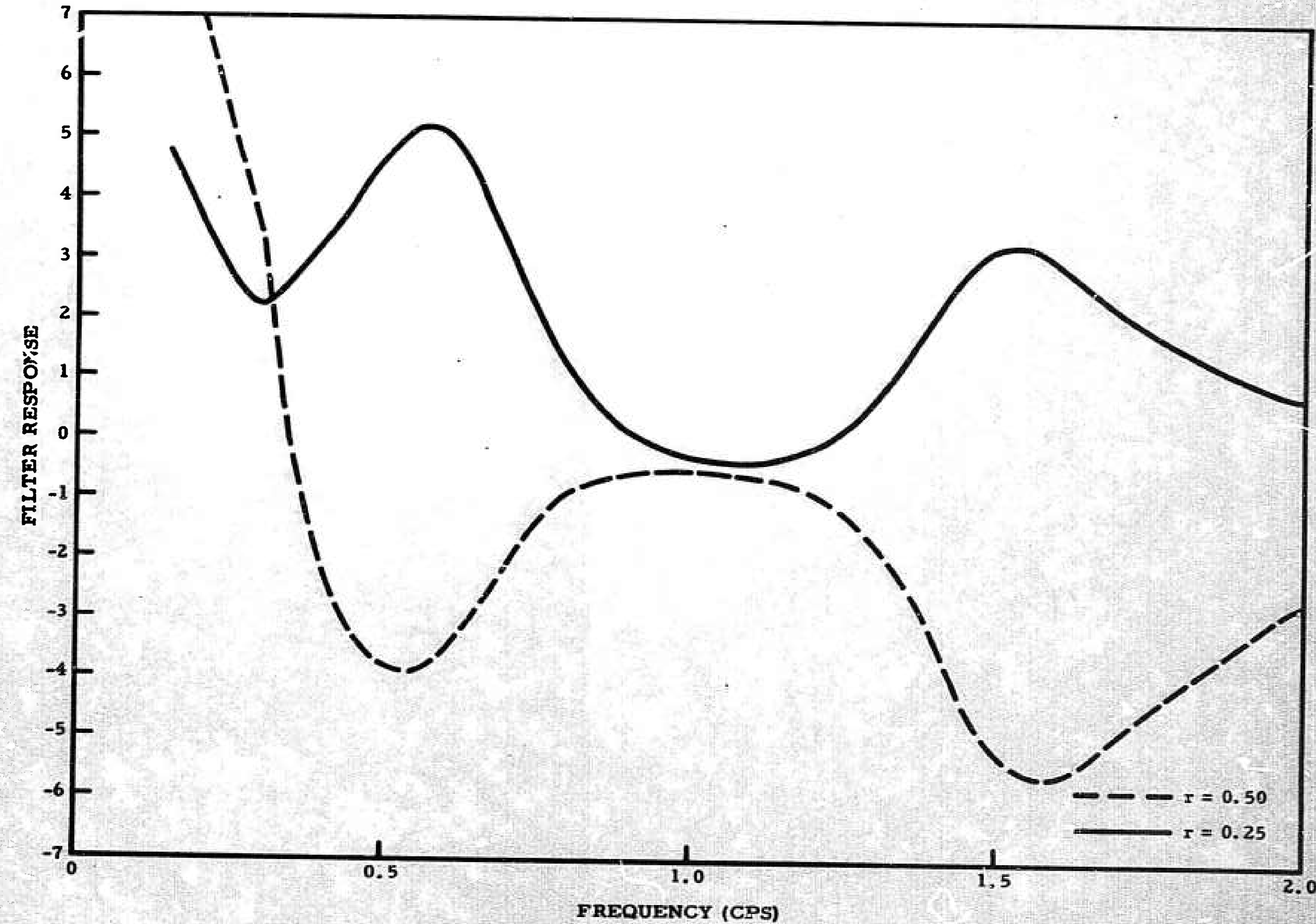

Figure 5.9. Computed Interpolation Filters for Case C (1 x Mode 1, 7 x Mode 2, 1 Percent Uncorrelated Noise on Horizontal Ring)

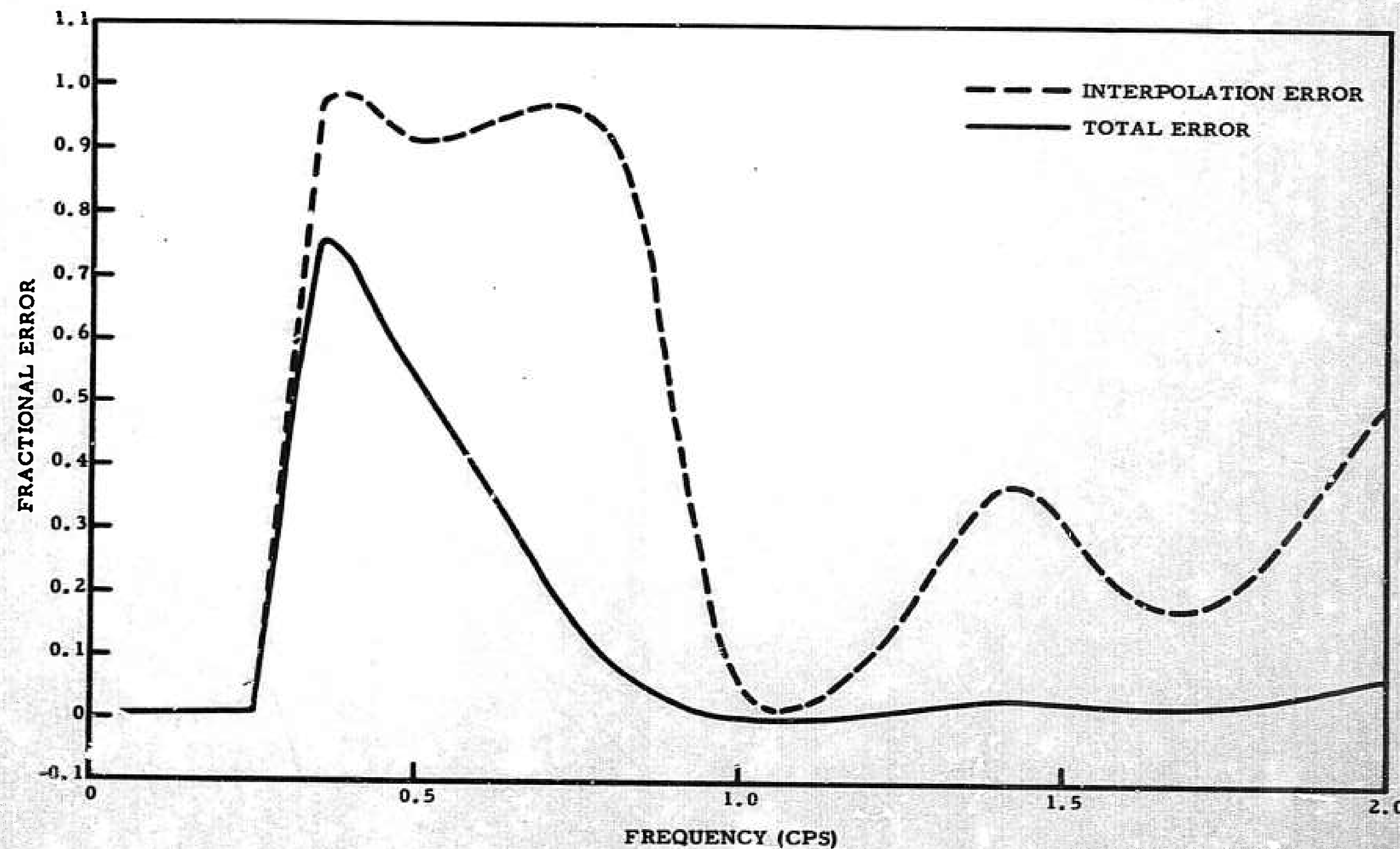

Figure 5.10. Interpolation Error and Total Error for Case C

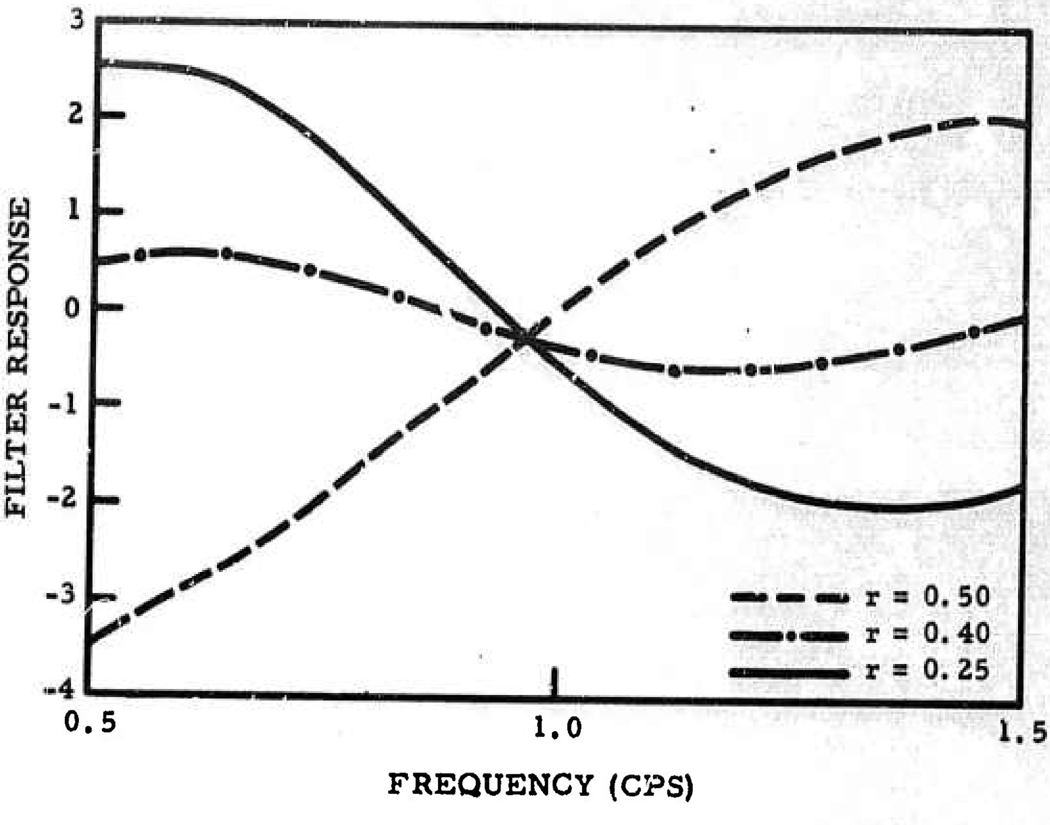

Figure 5.11. Computed Interpolation Filters for Case D (1 x Mode 1, 7 x Mode 2, 10 x Mode 3, 1 Percent Uncorrelated Noise on Horizontal Rings)

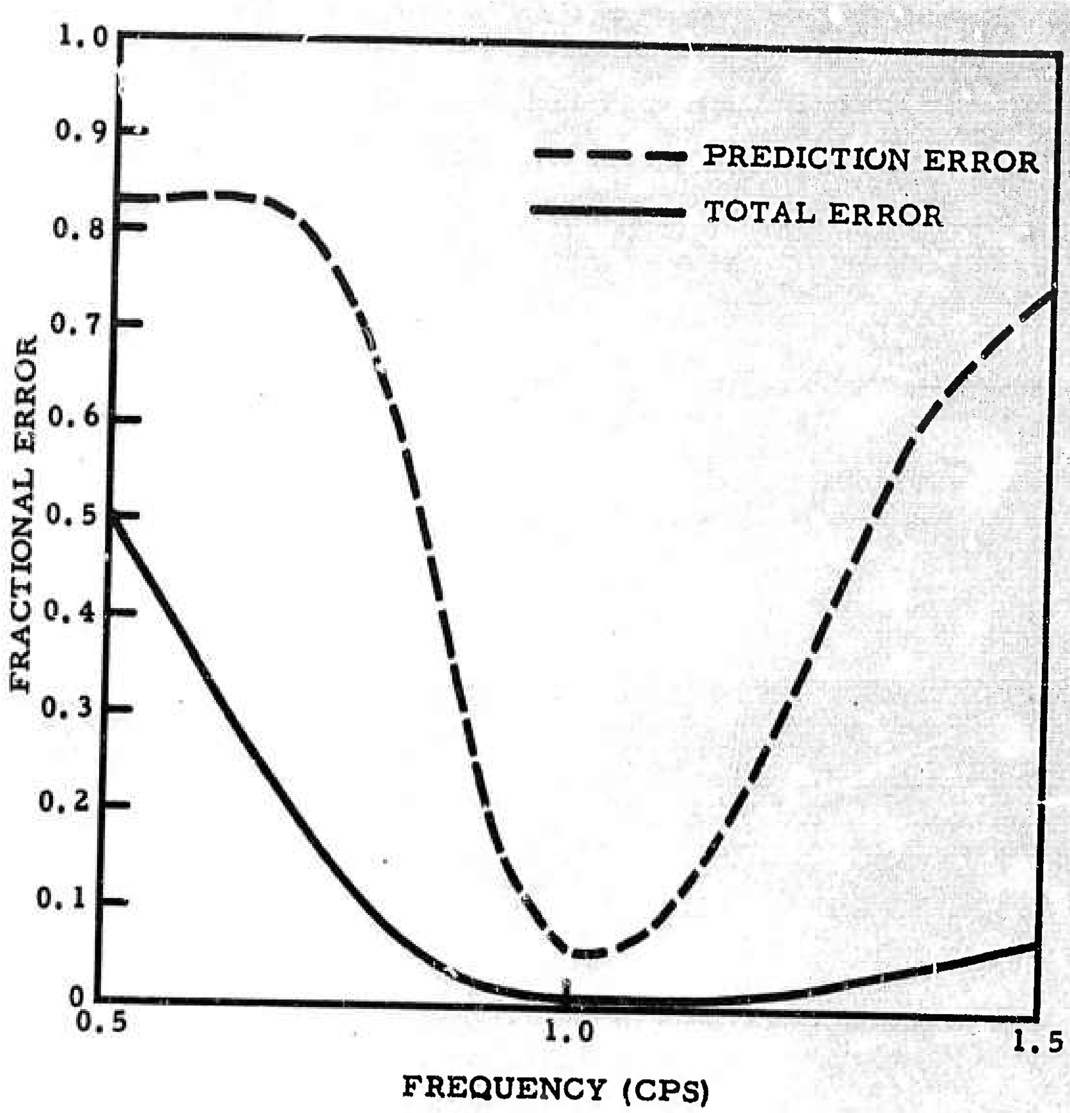

Figure 5.12. Interpolation Error and Total Error for Case D

SECTION VI

CONCLUSION

The results of this study point toward important future applications of horizontal-vertical interpolation arrays in the fields of teleseismic signal detection and oil exploration. Even in multimode isotropic Rayleigh waves, horizontal-vertical interpolation arrays have the theoretical capability of canceling almost all surface wave noise on a vertical trace while passing a vertical P-wave without distortion. In Section IV, we saw that good performance may be expected even if the P-wave is not quite vertical.

Additional study is required of the problem of optimizing array geometry. It appears that the overall dimensions of horizontal-vertical prediction arrays will in general be smaller than for vertical arrays with the same number of instruments (all of the arrays considered in Section V were less than 2 km in greatest dimension). In fact, were it not for the necessity of assuming the presence of uncorrelated noise on the predicting channels, eqs. (3.18) show that the best performance would be approached as the radii of the horizontal rings approach zero. The question of how best to choose the ring radii, however, has not been answered satisfactorily. The same is true of the question of choosing the number of seismometers and rotation angles for each ring. It is suggested that further studies be undertaken to determine array response for representative noise models, such as those in Section V. In order to aid in the understanding of the functioning of a given array, it may be helpful to prepare array response plots of the type described in Section IV B.

Finally, we must observe that all of our work here has one important shortcoming: it is not based upon real data. It is important now to procede from the theoretical stage of investigation to the experimental. The crucial question is not how well horizontal-vertical interpolation arrays work on paper — it is how well they work in the field.

SECTION VII

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